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Variational Inequalities of Perturbed Maximal Monotone Mapping with Applications

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Abstract—In this paper, we obtain some existence theorems of the variational inequalities associated to the perturbed maximal monotone mapping and discuss several surjectivity results for perturbed maximal monotone mappings.

Keywords—Variational inequality problem, T-pseudomonotone mapping, Maximal monotone mapping, Surjectivity.

1. INTRODUCTION AND PRELIMINARIES

Variational inequalities, introduced by Hartman, Stampacchia and Browder, have been developed rapidly for nearly thirty years. Variational inequality theory has become a rich source of inspiration in pure and applied mathematics, which has not only stimulated new and deep results in dealing with nonlinear partial differential equations, but has also provided us a unified and general framework for studying many problems arising in mechanics, physics, optimization and control, nonlinear programming, engineering sciences, etc.; see [1–4]. In 1988, Shih and Tan [5] got some existence results of variational inequalities for multivalued monotone mapping and obtained the surjectivity result for multivalued monotone mapping via the variational inequalities. Enlightened by [5], first we study the variational inequalities of sum of a maximal monotone mapping T and a T-pseudomonotone mapping T_0. Next, we discuss the surjectivity results of that. We differ from Shih and Tan [5] in proof techniques; the proof in [5] depends on the monotonicity of mappings, but the sum of T and T_0 is not monotone. For this reason, we cannot get our results from the methods in [5]. The surjectivity results in this paper answer affirmatively Browder's open question (see [6, p. 70]): let $T : E \to 2^{E^*}$ be a maximal monotone mapping, where E is a reflexive Banach space, E^* its dual space. Let T_0 be a bounded finitely continuous T-pseudomonotone mapping from E to E^*; suppose that $T + T_0$ is coercive. Is it then true that $T + T_0$ is surjective? There is no need for the separability hypotheses on E in this paper, so our surjectivity results are better than the corresponding results of Zhao [7–9].

In what follows, let E be a reflexive Banach space, E^* its dual space, $\langle .,. \rangle$, $\to$ and $\rightharpoonup$ stand for the pairing between E and E^*, strong and weak convergence of E, respectively. For a multivalued mapping $T : E \to 2^{E^*}$, we denote $D(T) = \{x \in E; T(x) \neq \emptyset\}$, $R(T) = \{w; \forall x \in D(T),$

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$w \in T(x)\}$, $G(T) = \{[x, w]; [x, w] \in E \times E^*,\ x \in D(T),\ w \in T(x)\}$. We first recall some notions which can be seen in [5,6,10,11].

DEFINITIONS. *A multivalued mapping $T : E \to 2^{E^*}$ is said to be quasi-bounded if for each $M > 0$ there exists $K(M) > 0$ such that whenever $[u, w]$ lies in the $G(T)$ and $\operatorname{Re}\langle w, u\rangle \le M\|u\|$, $\|u\| \le M$, then $\|w\| \le K(M)$. A multivalued mapping $T : E \to 2^{E^*}$ is said to be monotone if for any $x, y \in D(T)$, the inequality $\operatorname{Re}\langle w-v, x-y\rangle \ge 0$ holds for all $w \in T(x), v \in T(y)$. A monotone mapping T is said to be maximal monotone, if the inequality $\operatorname{Re}\langle w-v, x-y\rangle \ge 0\ (\forall\,[y, v] \in G(T))$ implies $x \in D(T)$ and $w \in T(x)$. A single-valued mapping $T_0 : E \to E^*$ is said to be T-pseudomonotone in Browder's sense [6], if for $\forall\{y_j\} \subset E$, $y_n \rightharpoonup y \in E$ and a bounded sequence $\{w_j\}$ with w_j in $T(y_j)$, suppose that $\overline{\lim}\operatorname{Re}\langle T_0(y_j), y_j - y\rangle \le 0$. Then $T_0(y_j) \rightharpoonup T_0(y)$ and $\operatorname{Re}\langle T_0(y_j), y_j - y\rangle \to 0$. Let K be a nonempty subset of E, $T : K \subset E \to 2^{E^*}$ be a multivalued mapping, then the variational inequality problem $\mathrm{VIP}(T, K)$ is to find $y_0 \in K, w_0 \in T(y_0)$ such that $\operatorname{Re}\langle w_0, x - y_0\rangle \ge 0$ for all $x \in K$.*

To state our theorems, we need the following results.

LEMMA 1. *Let F be a finite-dimensional Banach space, D be a nonempty closed convex subset of F. Suppose that $T : D \to 2^{F^*}$ is an upper semicontinuous multivalued mapping satisfying for each $x \in D$, $T(x)$ is a nonempty bounded closed convex subset of F^*, and there exists $x_0 \in D$ such that $\lim_{y\in D, \|y\|\to\infty} \inf_{w\in T(y)} \operatorname{Re}\langle w, y - x_0\rangle > 0$. Then $\mathrm{VIP}(T, D)$ has a solution.*

PROOF. For each $x \in D$, let

$$M(x) = \{y \in D;\ \text{there exists } w \in T(y) \text{ such that } \operatorname{Re}\langle w, x - y\rangle \ge 0\}.$$

First, we show that $\operatorname{co}\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^n M(x_i)$ for each $\{x_1, x_2, \ldots, x_n\} \subset D$. Indeed, if $z \in \operatorname{co}\{x_1, x_2, \ldots, x_n\}, z = \sum_{i=1}^n \lambda_i x_i$, where $\lambda_i \ge 0\,(i = 1, 2, \ldots, n)$ and $\sum_{i=1}^n \lambda_i = 1$, but $z \notin \bigcup_{i=1}^n M(x_i)$, then for all $w \in T(z)$, $\operatorname{Re}\langle w, x_i - z\rangle < 0$. Since also $0 = \operatorname{Re}\langle w, z - z\rangle = \operatorname{Re}\langle w - \sum_{i=1}^n \lambda_i x_i\rangle = \sum_{i=1}^n \lambda_i \operatorname{Re}\langle w, z - x_i\rangle > 0$, this contradiction implies that $\operatorname{co}\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^n M(x_i)$.

Next, we shall prove that $\overline{M(x_0)}$ is a compact subset of D. In fact, by conditions of Lemma 1, there exists a sufficiently large $\beta > 0$ such that for $\forall\, y \in D$ with $\|y\| > \beta$

$$\inf_{w\in T(y)} \operatorname{Re}\langle w, y - x_0\rangle > 0. \tag{1}$$

Take $B = \{y \in D; \|y\| \le \beta\}$, then B is a compact subset of D. Using (1), we know that $M(x_0) \subset B$, and therefore $\overline{M(x_0)}$ is a compact subset of D.

Finally, by the Ky Fan theorem (see [5]), we imply that $\bigcap_{x\in D} \overline{M(x)} \ne \emptyset$, hence there exists $y_0 \in \bigcap_{x\in D} \overline{M(x)}$, i.e., $y_0 \in \overline{M(x)}\,(\forall\, x \in D)$. It follows that there exists $\{y_n\} \subset M(x)$ such that $y_n \to y_0$, i.e., there exists $w_n \in T(y_n)$ such that $\operatorname{Re}\langle w_n, x - y_n\rangle \ge 0\,(n = 1, 2, \ldots)$. Since T is upper semicontinuous, we have that there exists $\{w_{n_j}\} \subset \{w_n\}$ such that $w_{n_j} \rightharpoonup w_0 \in T(y_0)$, thus for each $x \in D$, $\operatorname{Re}\langle w_0, x - y_0\rangle \ge 0$; that is, $\mathrm{VIP}(T, D)$ has a solution. ∎

Let K be a nonempty closed convex subset of E, $T : K \to 2^{E^*}$ satisfies the following condition:

(a) there exists $x_0 \in K$ such that $\lim_{y\in K, \|y\|\to\infty} \inf_{w\in T(y)} \operatorname{Re}\langle w, y - x_0\rangle > 0$.

We denote Λ by the family of all finite-dimensional subspace F of E, with F containing x_0 of condition (a), ordered by inclusion. Denote $K_F = K \cap F$. For each $F \in \Lambda$, we set

$$V_F = \{y \in K, \text{ there exists } w \in T(y) \text{ such that } \forall\, x \in K_F,\ \operatorname{Re}\langle w, x - y\rangle \ge 0\}$$

and denote $\overline{V_F}^w$ the weak closure of V_F.

LEMMA 2. *Let $T : K \to 2^{E^*}$ be a multivalued mapping with $T(x)$ a nonempty bounded closed convex subset of E^* for each $x \in K$. Suppose that T satisfies condition (a) and for each $F \in \Lambda$, $T : K_F \to 2^{E^*}$ is upper semicontinuous from the topology of F to the weak topology of E^*. Then $\bigcap_{F\in\Lambda} \overline{V_F}^w \neq \emptyset$.*

PROOF. For each $F \in \Lambda$, let $j_F : F \to E$ be the inclusion map of F into E, and $j_F^* : E^* \to F^*$ be the dual projection map of j_F from E^* onto F^*. Consider the operator $T_F = j_F^* T j_F : K_F \to 2^{F^*}$. By conditions of Lemma 2, we imply that T_F is upper semicontinuous from K_F to 2^{F^*}, and for each $x \in K_F$, $T_F(x)$ is a nonempty bounded closed convex subset of F^*. It follows from condition (a) that

$$\begin{aligned}\lim_{y\in K_F,\|y\|\to\infty} \left\{ \inf_{w\in T_F(y)} \operatorname{Re}\langle w, y - x_0\rangle \right\} &= \lim_{y\in K_F,\|y\|\to\infty} \left\{ \inf_{w\in T(y)} \operatorname{Re}\langle j_F^* w, y - x_0\rangle \right\} \\ &= \lim_{y\in K_F,\|y\|\to\infty} \left\{ \inf_{w\in T(y)} \operatorname{Re}\langle w, y - x_0\rangle \right\} > 0.\end{aligned}$$

That is, T_F satisfies all conditions of Lemma 1. By Lemma 1, there exist $y_0 \in K_F$, $w_F \in T_F(y_0)$ such that $\operatorname{Re}\langle w_F, x - y_0\rangle \geq 0$, for each $x \in K$. Thus, there exists $w_0 \in T(y_0)$ such that $w_F = j_F^* w_0$. It follows that $\forall\, x \in K_F$, $\operatorname{Re}\langle w_0, x - y_0\rangle = \operatorname{Re}\langle j_F^* w_0, x - y_0\rangle = \operatorname{Re}\langle w_F, x - y_0\rangle \geq 0$; that is, $V_F \neq \emptyset$ ($\forall F \in \Lambda$). We shall show that $\overline{V_F}^w$ is a weakly compact subset of K. Indeed, by condition (a), there exists a sufficiently large $\beta > 0$ such that

$$\inf_{w\in T(y)} \operatorname{Re}\langle w, y - x_0\rangle > 0, \qquad \forall\, x \in K, \quad \|y\| > \beta. \tag{2}$$

Take $B = \{y \in K; \|y\| \leq \beta\}$. It is easy to prove from (2) that $V_F \subset B$. Notice that B is a bounded closed subset of the reflexive Banach E, hence B is weakly compact, and thus $\overline{V_F}^w \subset B$; that is, $\overline{V_F}^w$ is a weakly compact subset of K.

Denote $I = \{1, 2, \ldots, n\}$, and for each $F_i \in \Lambda$, $i \in I$, set $\hat{F} = \{\sum_{i=1}^n t_i x_i;\ \forall x_i \in F_i,\ t_i \in R,\ i \in I\}$. We have that $\hat{F} \in \Lambda$ and $F_i \subset \hat{F}$ for each $i \in I$, therefore $V_{\hat{F}} \neq \emptyset$; that is, there exist $y \in K$, $w \in T(y)$, such that for all $x \in K_{F_i}, i \in I$, $\operatorname{Re}\langle w, x - y\rangle \geq 0$. Hence $y \in V_{F_i} \subset \overline{V_{F_i}}^w$, and it is implied that $\bigcap_{i=1}^n \overline{V_{F_i}}^w \neq \emptyset$, i.e., $\{\overline{V_F}^w\}_{F\in\Lambda}$ satisfies the finite intersection property, and consequently, we obtain $\bigcap_{F\in\Lambda} \overline{V_F}^w \neq \emptyset$. ∎

2. MAIN RESULTS

In this section, let K be a nonempty closed convex subset of the reflexive Banach space E.

THEOREM 1. *Let $T : E \to 2^{E^*}$ be a bounded maximal monotone mapping such that $D(T) = E$ and $T_0 : K \to E^*$ be a finitely continuous T-pseudomonotone mapping. Assume there exists $x_0 \in K$ such that the following condition (b) is satisfied:*

(b) $\lim_{y\in K,\|y\|\to\infty} \inf_{w\in T(y)} \operatorname{Re}\langle w + T_0(y), y - x_0\rangle > 0.$

Then $\mathrm{VIP}(T + T_0, K)$ *has a solution.*

PROOF. By Proposition 8 of [11], we know that $T(x)$ is a nonempty bounded closed convex subset of E^* for each $x \in E$, and T is upper semicontinuous from the topology of each finite-dimensional subspace of E to the weak topology on E^*. For each $F \in \Lambda$, we set

$$\mathcal{V}_F = \{y \in K;\ \text{there exist } w \in T(y), \text{ for each } x \in K_F,\ \operatorname{Re}\langle w + T_0(y), x - y\rangle \geq 0\}.$$

It follows from condition (b) of Theorem 1 that $T + T_0$ satisfies the condition (a). Hence, by Lemma 2, we have that there exists $y_0 \in \bigcap_{F\in\Lambda} \overline{\mathcal{V}_F}^w$. For each $x \in K$, we can find $F_0 \in \Lambda$ such that $x, y_0 \in F_0$, i.e., $x, y_0 \in K_{F_0}$. Since $y_0 \in \overline{\mathcal{V}_{F_0}}^w$, then there exists a subsequence $\{y_n\}$ in $\mathcal{V}_{F_0}$ such that $y_n \rightharpoonup y_0$, hence there exist $w_n \in T(y_n)$ such that

$$\operatorname{Re}\langle w_n + T_0(y_n), x - y_n\rangle \geq 0 \tag{3}$$

and

$$\operatorname{Re}\langle w_n + T_0(y_n), y_0 - y_n\rangle \geq 0. \tag{4}$$

Taking $w_1 \in T(y_0)$ and using (4), we have that $\operatorname{Re}\langle w_n - w_1, y_0 - y_n\rangle + \operatorname{Re}\langle w_1, y_0 - y_n\rangle + \operatorname{Re}\langle T_0(y_n), y_0 - y_n\rangle \geq 0$. By the monotonicity of T, we have that $\operatorname{Re}\langle w_1, y_0 - y_n\rangle + \operatorname{Re}\langle T_0(y_n), y_0 - y_n\rangle \geq 0$. Therefore

$$\overline{\lim}\ \operatorname{Re}\langle T_0(y_n), y_n - y_0\rangle \leq 0. \tag{5}$$

The boundedness of $\{y_n\}$ and T implies that $\{w_n\}$ is bounded. Since T_0 is T-pseudomonotone, it follows from (5) that $T_0(y_n) \rightharpoonup T_0(y_0)$ and

$$\lim\ \operatorname{Re}\langle T_0(y_n), y_n - y_0\rangle = 0. \tag{6}$$

Using the boundedness of $\{w_n\}$ and (4), (6), we know that there exist $\{w_j\} \subset \{w_n\}$ such that $w_j \rightharpoonup w_0$ and $\overline{\lim}\ \operatorname{Re}\langle w_j, y_j - y_0\rangle \leq 0$. From Proposition 2 of [11], we have that the maximal monotone mapping T is generalized pseudomonotone (the definition can be seen in [11]), hence $w_0 \in T(y_0)$ and

$$\lim\ \operatorname{Re}\langle w_j, y_j - y_0\rangle = 0. \tag{7}$$

It is implied from (3), (6), (7) that there exist $y_0 \in K$, $w_0 \in T(y_0)$, such that

$$\operatorname{Re}\langle w_0 + T_0(y_0), x - y_0\rangle \geq 0, \qquad \forall\, x \in K, \tag{8}$$

i.e., VIP$(T + T_0, K)$ has a solution. ∎

THEOREM 2. *Let $T : E \to 2^{E^*}$ be a maximal monotone mapping (not necessarily bounded) with $D(T) = E$ and $T_0 : K \to E^*$ be a bounded finitely continuous T-pseudomonotone mapping. Assume $0 \in K$ and the following condition (c) holds:*

(c) $\lim_{y\in K, \|y\|\to\infty} \inf_{w\in T(y)} \operatorname{Re}\langle w + T_0(y), y\rangle > 0$.

Then VIP$(T + T_0, K)$ *has a solution.*

PROOF. Taking $x_0 = 0$, we know from the condition (c) that $T + T_0$ satisfies the condition (a). For each $F \in \Lambda$, define $\mathcal{V}_F$ as above in Theorem 1, and similar to proof of Theorem 1, we have that there exists $y_0 \in \bigcap_{F\in\Lambda} \overline{\mathcal{V}_F}^w$. For each $x \in K$, we can find $F_0 \in \Lambda$ such that $x, y_0 \in F_0$, i.e., $x, y_0 \in K_{F_0}$. Since $y_0 \in \overline{\mathcal{V}_{F_0}}^w$, then there exists a subsequence $\{y_n\}$ in $\mathcal{V}_{F_0}$ such that $y_n \rightharpoonup y_0$; hence there exist $w_n \in T(y_n)$ such that the inequalities (3)–(5) hold. We shall prove that $\{w_n\}$ is bounded. Indeed, we imply by $0 \in K$ and $0 \in F$ that $0 \in K_F$, and therefore $\operatorname{Re}\langle w_n + T(y_n), 0 - y_n\rangle \geq 0$; that is, $\operatorname{Re}\langle w_n, y_n\rangle \leq -\operatorname{Re}\langle T_0(y_n), y_n\rangle$. From [10], we know that T is quasi-bounded, and it follows from the above inequality and the boundedness of $\{y_n\}$ and T_0 that $\{w_n\}$ is bounded. Analogous to proof of Theorem 1, we know that the inequalities (6)–(8) hold, i.e., VIP$(T + T_0, K)$ has a solution. ∎

COROLLARY 1. *Let $T : E \to 2^{E^*}$ be a bounded maximal monotone mapping such that $D(T) = E$ and $T_0 : K \to E^*$ is a T-pseudomonotone mapping. Assume that there exists $x_0 \in K$ such that*

$$\lim_{y\in K, \|y\|\to\infty}\ \inf_{w\in T(y)} \frac{\operatorname{Re}\langle w + T_0(y), y - x_0\rangle}{\|y\|} = +\infty.$$

Then for each $f \in E^*$, there exist $y_0 \in K, w_0 \in T(y_0)$, such that

$$\operatorname{Re}\langle w_0 + T_0(y_0) - f, x - y_0\rangle \geq 0, \qquad \forall\, x \in K.$$

PROOF. Since

$$\begin{aligned}
&\lim_{y\in K, \|y\|\to\infty}\ \inf_{w\in T(y)} \frac{\operatorname{Re}\langle w + T_0(y) - f,\ y - x_0\rangle}{\|y\|} \\
&\qquad = \lim_{y\in K, \|y\|\to\infty}\ \inf_{w\in T(y)} \left(\frac{\operatorname{Re}\langle w + T_0(y),\ y - x_0\rangle}{\|y\|} - \|f\|\right) = +\infty,
\end{aligned}$$

we set $\mathcal{T}(y) = T(y) - f$, $(\forall y \in K)$, then $\mathcal{T} + T_0$ satisfies the conditions of Theorem 1. By Theorem 1, VIP$(\mathcal{T} + T_0, K)$ has a solution. Hence, there exists $w_0 \in T(y_0)$ such that $\overline{w} = w_0 - f$ and for each $x \in K$, $\operatorname{Re}\langle w_0 + T_0(y_0) - f, x - y_0\rangle \geq 0$. ∎

COROLLARY 2. *Let $T : E \to 2^{E^*}$ be a maximal monotone mapping (not necessarily bounded) with $D(T) = E$, $T_0 : K \to E^*$ be a bounded finitely continuous T-pseudomonotone mapping and $0 \in K$. Suppose that $T + T_0$ is coercive on K, i.e., there exists a real function $C(r)$ on R^+, $\lim_{r\to+\infty} C(r) = +\infty$, such that for $\forall y \in K$, $w \in T(y)$, $\operatorname{Re}\langle w+T_0(y), y\rangle \ge C(\|y\|)\,\|y\|$. Then for each $f \in E^*$, there exist $y_0 \in K$, $w_0 \in T(y_0)$, such that $\forall x \in K$, $\operatorname{Re}\langle w_0 + T_0(y_0) - f, x - y_0\rangle \ge 0$.*

PROOF. From the coerciveness of $T + T_0$,

$$\lim_{y\in K,\|y\|\to\infty}\ \inf_{w\in T(y)} \frac{\operatorname{Re}\langle w + T_0(y),\, y\rangle}{\|y\|} = +\infty.$$

Similar to the proof of Corollary 1, we can know that the conclusion of Corollary 2 is fulfilled. ∎

From Corollary 1, we can get easily the following two surjectivity results.

COROLLARY 3. *Let $T : E \to 2^{E^*}$ be a bounded maximal monotone mapping with $D(T) = E$, $T_0 : E \to E^*$ be a finitely continuous T-pseudomonotone mapping. Suppose that there exists $x_0 \in E$ such that*

$$\lim_{y\in E,\|y\|\to\infty}\ \inf_{w\in T(y)} \frac{\operatorname{Re}\langle w + T_0(y), y - x_0\rangle}{\|y\|} = +\infty.$$

Then $T + T_0$ is surjective, that is, $R(T + T_0) = E^*$.

COROLLARY 4. *Let $T : E \to 2^{E^*}$ be a maximal monotone mapping (not necessarily bounded) with $D(T) = E$, $T_0 : E \to E^*$ be a bounded finitely continuous T-pseudomonotone mapping. Suppose that $T + T_0$ is coercive, then $T + T_0$ is surjective.*

THEOREM 3. *Let $T : E \to 2^{E^*}$ be a maximal monotone mapping (not necessarily bounded) with $0 \in D(T)$, $T_0 : E \to E^*$ be a quasi-bounded finitely continuous T-pseudomonotone mapping. Assume that there exists $x_0 \in E$ such that*

$$\lim_{y\in E,\|y\|\to\infty} \frac{\operatorname{Re}\langle T_0(y),\, y - x_0\rangle}{\|y\|} = +\infty.$$

Then $T + T_0$ is surjective.

PROOF. We may assume without loss of generality that $[0, 0] \in G(T)$. For $\forall\varepsilon > 0$, we consider the generalized *Yosido* approximations of T : $T_\varepsilon = (T^{-1} + \varepsilon J^{-1})^{-1}$, where J is the duality mapping defined by $J(x) = \{f \in E^*; \langle f, x\rangle = \|x\|^2 \text{ and } \|f\| = \|x\|\}$ $(\forall x \in E)$. By Proposition 12 in [10], we have that $T_\varepsilon : E \to 2^{E^*}$ is a bounded maximal monotone mapping and $D(T_\varepsilon) = E$. Let $\hat{w} \in T_\varepsilon(x_0)$, the monotonicity of T_ε implies that $\forall w_\varepsilon \in T_\varepsilon(y)$, $\operatorname{Re}\langle w_\varepsilon - \hat{w}, y - x_0\rangle \ge 0$, and so

$$\begin{aligned}\lim_{y\in E,\|y\|\to\infty}\ \inf_{w_\varepsilon\in T_\varepsilon(y)} &\frac{\operatorname{Re}\langle w_\varepsilon + T_0(y),\, y - x_0\rangle}{\|y\|}\\ &\ge \lim_{y\in E,\|y\|\to\infty} \frac{\operatorname{Re}\{\langle T_0(y),\, y - x_0\rangle + \langle \hat{w},\, y - x_0\rangle\}}{\|y\|} = +\infty. \qquad (9)\end{aligned}$$

By Corollary 3, we have that $R(T_\varepsilon + T_0) = E^*$.

We take $f_0 \in X^*$ arbitrarily, then there exist $y_\varepsilon \in E$, $w_\varepsilon \in T_\varepsilon(y_\varepsilon)$ such that

$$f_0 = w_\varepsilon + T_0(y_\varepsilon). \qquad (10)$$

From $\operatorname{Re}\langle w_\varepsilon, y_\varepsilon\rangle + \operatorname{Re}\langle T_0(y_\varepsilon), y_\varepsilon\rangle = \operatorname{Re}\langle f_0, y_\varepsilon\rangle$ and $[0, 0] \in G(T_\varepsilon)$, we have $\operatorname{Re}\langle w_\varepsilon, y_\varepsilon\rangle \ge 0$, hence $\operatorname{Re}\langle T_0(y_\varepsilon), y_\varepsilon\rangle \le \operatorname{Re}\langle f_0, y_\varepsilon\rangle \le \|f_0\|\cdot\|y_\varepsilon\|$. By (9), $\{y_\varepsilon\}$ is uniformly bounded. Hence $\{T_0(y_\varepsilon)\}$ is uniformly bounded by the quasi-boundedness of T_0, and $\{w_\varepsilon\}$ is bounded from (10). Let $\varepsilon_0 > 0$, then $\exists M > 0$ such that $\|y_\varepsilon\| \le M$, $\|w_\varepsilon\| \le M$, $\|T_0(y_\varepsilon)\| \le M$ $(\forall\varepsilon, 0 < \varepsilon < \varepsilon_0)$. It is implied from the definition of T_ε that there exists $x_\varepsilon \in D(T)$ such that $w_\varepsilon \in T(x_\varepsilon)$, and hence, $\varepsilon w_\varepsilon \in J(y_\varepsilon - x_\varepsilon)$, that is $\|y_\varepsilon - x_\varepsilon\| = \varepsilon\|w_\varepsilon\| \le \varepsilon M \le \varepsilon_0 M$, and hence $\|x_\varepsilon\| \le (\varepsilon_0 + 1)M$.

We may find a sequence of values $\varepsilon_j \to 0^+$ such that has the notation

$$w_j = w_{\varepsilon_j}, \quad T_0(y_j) = T_0\left(y_{\varepsilon_j}\right), \quad y_j = y_{\varepsilon_j}, \quad x_j = x_{\varepsilon_j}.$$

The sequences $\{y_j\}$ and $\{x_j\}$ converge weakly to a common limit $y_0 \in E$, while $w_j \rightharpoonup w_0$ and $T_0(y_j) \rightharpoonup T_0(y_0)$ with

$$w_0 + T_0(y_0) = f_0. \tag{11}$$

The equations (10), (11) yield that

$$\operatorname{Re}\langle (w_j + T_0(y_j)) - (w_0 + T_0(y_0)), x_j - y_0 \rangle = 0. \tag{12}$$

The generalized pseudomonotonicity of T and Proposition 2 of [10] yield that $\underline{\lim} \operatorname{Re}\langle w_j, x_j - y_0\rangle \geq 0$ and consequently, $\overline{\lim} \operatorname{Re}\langle T_0(y_j), x_j - y_0\rangle \leq 0$. Since also T_0 is T-pseudomonotone, we have that $\lim \operatorname{Re}\langle T_0(y_j), x_j - y_0\rangle = 0$. By the equation (12), we know that $\lim \operatorname{Re}\langle w_j, x_j - y_0\rangle = 0$. It is implied from the generalized pseudomonotonicity of T that $w_0 \in T(y_0)$, thus $w_0 + T_0(y_0) \in (T + T_0)(y_0)$; that is, $f_0 \in (T + T_0)(y_0)$. Hence $T + T_0$ is surjective. ∎

REMARK 1. In Corollary 4, we answer affirmatively Browder's question (see Section 1) on the condition that $D(T) = E$. Browder's question has already been answered affirmatively in [7] if there exists an injective approximation scheme for (E, E^*) and T is singlevalued. If E is a separable and reflexive Banach space, it can be given an injective approximation scheme for (E, E^*). There is no need for the separability hypotheses on E in Corollary 4, so our surjectivity results are better than the corresponding ones of Zhao [7–9]. If the condition that $T + T_0$ is coercive is turned into T_0 being coercive, then $D(T) = E$ can be weakened to $D(T) \neq \emptyset$; in fact, that is Theorem 3.

REMARK 2. In [12], various refinements of older results involving pseudomonotone perturbation of maximal monotone operators are given. These results have as special cases various surjectivity results which can also be obtained via variational inequalities.

REFERENCES

1. P. Hartman and G. Stampacchia, On some nonlinear elliptic differential functional equation, *Acta Math.* **115**, 271–310, (1966).
2. F.E. Browder, The fixed point theory of multivalued mappings in topological vector spaces, *Math. Ann.* **177**, 283–301, (1968).
3. D. Kinderleher and G. Stampaccia, *An Introduction to Variational Inequalities*, Academic Press, New York, (1980).
4. C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities*, John Wiley & Sons, New York, (1984).
5. M.H. Shih and K.K. Tan, Browder-Hartman-Stampacchia variational inequalities for multivalued monotone operators, *J. Math. Anal. Appl.* **134**, 431–440, (1988).
6. F.E. Browder, Nonlinear functional analysis in mathematical developments arising from Hilbert problems, Vol. 1, *Proc. Symp. in Pure Math. of Amer. Soc.* **28**, 68–73, (1976).
7. Y.C. Zhao, On the topological degree for the sum of two maximal monotone operator and generalized pseudomonotone operator, *Chin. Ann. of Math.* **4B** (2), 241–253, (1983).
8. Y.C. Zhao, Surjectivity of perturbed maximal monotone mapping, *Kexue Tongbao* **29** (7), 857–860, (1984).
9. Y.C. Zhao, On a surjectivity for the sum of two mappings of monotone type, *Chin. Ann. of Math.* **6B** (2), 471–480, (1985).
10. E. Zeilder, *Nonlinear Functional Analysis and its Applications*, **II/B**, Springer-Verlag, New York, (1990).
11. F.E. Browder and P. Hess, Nonlinear mappings of monotone type in Banach space, *J. Funct. Anal.* **11**, 251–294, (1972).
12. Z. Guan and A.G. Kartsatos, Ranges of perturbed maximal monotone and m-accretive operators in Banach spaces, *Trans. Amer. Math. Soc.* **347**, 2047–2435, (1995).